Recursive Powers Involving the Imaginary *I*

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Complex numbers are familiar to all high-school advanced mathematics students primarily because of their role in the solution to the general quadratic. Whether the students follow it up with the subsequent study of the general cubic or quartic is doubtful and, in fact, too bad. What is presented in this article, though, is a family of sequences of complex numbers that yield some interesting geometric results, with data that can be easily computed using a powerful computer algebra system like MAPLE or a graphing calculator, such as the TI-83.

Fundamentals

By definition, a complex number z has the standard form $z = a + bi$, where $a, b \in \Re$ and $i = \sqrt{-1}$. The number z corresponds uniquely to the point (a, b) in the Cartesian plane. Apart from the usual complex arithmetic, the value of a complex z raised to a complex w power is

$$z^w \quad = e^{\ln(z^w)} = e^{w\ln(z)}$$

and the logarithm function ln is defined on a complex argument by (Figure 1)

$$\ln(z) = \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta}) = \ln r + i\theta.$$

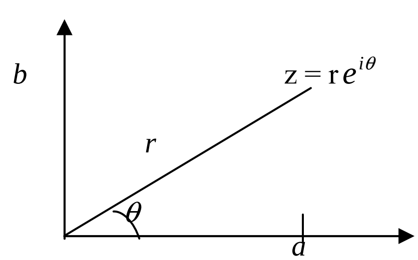

Figure 1

Euler's famous formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for any real θ allows us to replace an exponential raised to a complex power by a complex number in standard form. This will provide useful representations in what is soon to follow.

Suppose, for example, we consider ln(i). In polar form we have $i = 1\, e^{i\pi/2}$, since i is located on the y-axis at (0, 1), so

$$\ln(i) = \ln(1) + i\pi/2 = i\pi/2.$$

Students can verify this on their TI-83, for with the calculator set to complex $a + bi$ mode, the expression ln(i) is evaluated as 1.570796327i. Likewise, ln(-i) = -$i\pi$/2. A more interesting result is that the power i^i is in fact a real number, for

$$i^i = e^{i\ln(i)} = e^{-\pi/2}$$

and the TI-83 would evaluate i^i as 0.2078795764.

We also find $(-i)^{-i} = e^{-i\ln(-i)} = e^{-\pi/2}$.

The Main Sequence

The numbers i, i^i, i^{i^i}, ... form a sequence $\{z_n\}$ that yields some fascinating results. It is more convenient to define the sequence recursively, by

$$z_1 = \mathrm{i}, \quad z_{n+1} = i^{z_n} \quad \text{for all } n \geq 1.$$

One can rapidly convert each z_n into standard form on the TI-83 by appropriate use of the ANS key. To this end, first enter i,

i ENTER

followed by the recursive step

i ^ ANS ENTER

which immediately gives the value for i^i of 0.2078795764. Then by simply pushing ENTER in succession gives all later values z_n of the sequence. For instance, after 65 such iterations the calculator screen shows

0.4378664822 + 0.3603214496i
0.4387054563 + 0.3604591076i
0.4381352924 + 0.3609588861i
0.4381143187 + 0.3602833908i
0.4385913154 + 0.3606514286i

where these are the values for z_{61}, z_{62}, z_{63}, z_{64}, z_{65} respectively. The sequence is apparently converging to a point approximated by 0.4382 + 0.3605i.

Alternately let us write each z_n in standard form

$$z_1 = i = 0 + 1i = a_1 + b_1 i$$
$$z_2 = i^i = .2078795764 = a_2 + b_2 i$$
$$\vdots$$
$$z_{n+1} = i^{z_n} = a_{n+1} + b_{n+1} i.$$

Expanding further on this last equality gives

$$\begin{aligned} i^{z_n} = i^{a_n+b_n i} = e^{(a_n+b_n i)\ln i} &= e^{(a_n+b_n i)i\pi/2} \\ &= e^{-b_n\pi/2}\, e^{ia_n\pi/2} \\ &= e^{-b_n\pi/2}\,[\cos\, a_n\pi/2 + \mathrm{i}\cdot\sin\, a_n\pi/2] \end{aligned}$$

so $$a_{n+1} = e^{-b_n\pi/2}\cos\, a_n\pi/2 \quad \text{and} \quad b_{n+1} = e^{-b_n\pi/2}\sin\, a_n\pi/2. \qquad (1)$$

These relationships can be programmed in either MAPLE, which possesses great precision and speed, or on the TI-83, using the following code:

```
DISP "number terms = "
PROMPT  n
0 → a
π/2 → p
1 → b
FOR (k,1,n,1)
   a→ c
   e^(-bp) → d
   d*cos(cp) → a
   d*sin(cp) → b
END
DISP "after term ", n
DISP "value is ", a
DISP  b
STOP
```

Executing this program with $n = 400$ produces the output $A = 0.4382829367$ and $B = 0.3605924719$ where these values are interpreted as the limits of the two sequences $\{a_n\}$ and $\{b_n\}$, respectively. Applying limits to both sides of the equations in (1) gives

$$e^{-B\pi/2}\cos(A\pi/2) = A \quad \text{and} \quad e^{-B\pi/2}\sin(A\pi/2) = B \qquad (2)$$

or, after rearranging,

$$B = A\tan(A\pi/2).$$

Substituting this latter equation into (2) and solving for $\cos(A\frac{\pi}{2})$ yields

$$Ae^{A\frac{\pi}{2}\tan(A\frac{\pi}{2})} = \cos(A\frac{\pi}{2}).$$

If we define and graph the corresponding two functions on the TI-83 (Figure 2) then

$$y_1(x) = xe^{x\frac{\pi}{2}\tan(x\frac{\pi}{2})}$$

$$y_2(x) = \cos(x\frac{\pi}{2})$$

the point of intersection is the noted limit of $\{z_n\}$.

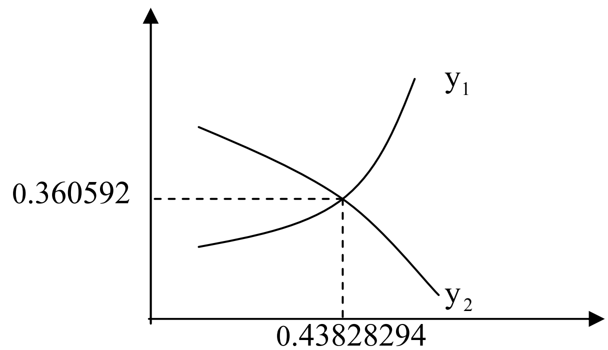

Figure 2

Some Generalizations

It would be a natural consideration to examine the sequence whose first terms are

$$-i,\ (-i)^{(-i)},\ (-i)^{(-i)^{(-i)}}.$$

Evaluating these terms on the TI-83 using the ANS key as before, the terms converge to 0.4382 - 0.3605i, which we note is the conjugate of the earlier limit. This can be verified rigorously by making use of the same method as before. We note

$$\begin{aligned} z_{n+1} = (-i)^{z_n} = (-i)^{c_n+d_n i} &= e^{(c_n+d_n i)\ln(-i)} \\ &= e^{(c_n+d_n i)(-i\pi/2)} \\ &= e^{d_n\pi/2}[\cos(-\pi c_n/2) + i\cdot\sin(-\pi c_n/2)] \\ &= e^{d_n\pi/2}[\cos(\pi c_n/2) - i\cdot\sin(\pi c_n/2)] \end{aligned}$$

where this last equality follows because cosine is an even function and sine is an odd function. Hence

$$c_{n+1} = e^{d_n\pi/2}\cos(\pi c_n/2) \quad \text{and} \quad d_{n+1} = -e^{d_n\pi/2}\sin(\pi c_n/2).$$

Since $(-i)^{(-i)} = i^i$ then $a_2 + b_2 i = c_2 + d_2 i$ with $b_2 = d_2 = 0$ and $a_2 = c_2 =$

0.2078795764. Thus $c_3 = e^{d_2\pi/2}\cos(\pi c_2/2) = \cos(\pi a_2/2) = a_3$ and $d_3 = -e^{d_2\pi/2}\sin(\pi c_2/2) = -\sin(\pi a_2/2) = -b_3$. This process continues indefinitely with $c_n = a_n$ and $d_n = -b_n$ for all $n \geq 2$ and this justifies the conjugacy of the limit.

Instead of looking at the repeated powers of i or $-i$, let's consider a more general setting of

$$w,\ w^w,\ w^{w^w},\ \ldots$$

where w is a complex number located on the unit circle (Figure 3). If we set $z_1 = w$, $z_{n+1} = w^{z_n}$ for all $n \geq 1$, and putting each z_n in standard form by $z_n = a_n + b_n i$

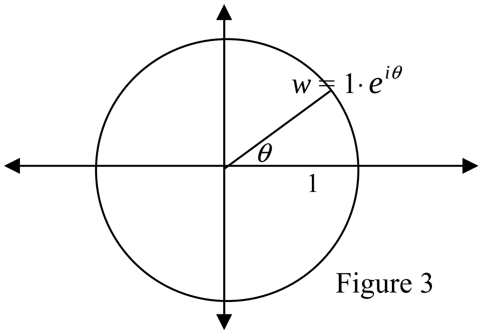

Figure 3

we would be able to develop the system of equations

$$a_{n+1} = e^{-b_n\theta}\cos(\theta a_n) \quad \text{and} \quad b_{n+1} = e^{-b_n\theta}\sin(\theta a_n)$$

and if each of these sequences converge, say $a_n \to \alpha$, $b_n \to \beta$, then these limits would be related by the familiar tangent equation

$$\beta = \alpha\tan(\theta\alpha).$$

As it turns out though, these sequences don't always converge! It is of interest to determine which ones do converge and which ones don't. This leads us to define a complex valued function *f* that maps each point *w* on the unit circle to the unique point in the plane, if one exists, that serves as the limit to the sequence

$$w,\ w^w,\ w^{w^w},\ \ldots .$$

Another way to say and represent this is

$$f(w) = w^{w^{\cdot^{\cdot^{\cdot}}}}$$

where the infinite exponential is actually the limit of the sequence of finite exponentials. Now we have previously determined the two function values

$$f(i) = (0.4382829367\ldots,\ 0.3605924718\ldots)$$
$$f(-i) = (0.4382829367\ldots,\ -0.3605924718\ldots)$$

and of course it is obvious that $f(1) = (1, 0)$. For all other *w* on the unit circle, one can simply run the recursive scheme with ANS on the TI-83 to determine the nature of $f(w)$. Thus, the sequence of steps

w	ENTER
w ^ ANS	ENTER
	ENTER
	⋮

with a sufficient number of ENTERS will approximate $f(w)$. Some selected values of $f(w)$ are as follows:

w	$f(w)$
$0.8 + 0.6i$	(0.7140, 0.3533)
$0.6 + 0.8i$	(0.6004, 0.3737)
$0.3 + \sqrt{.91}\, i$	(0.5026, 0.3713)

continued

$- 0.1 + \sqrt{.99}\, i$ (0.4205, 0.3564)
$- 0.35 + \sqrt{.8775}\, i$ (0.3813, 0.3449)

Symmetry exists when *w* and $\overline{w}$ are complex conjugates, for

$$f(\overline{w}) = \overline{f(w)}$$

as was exemplified when $w = i$. These results can be depicted graphically in Figure 4.

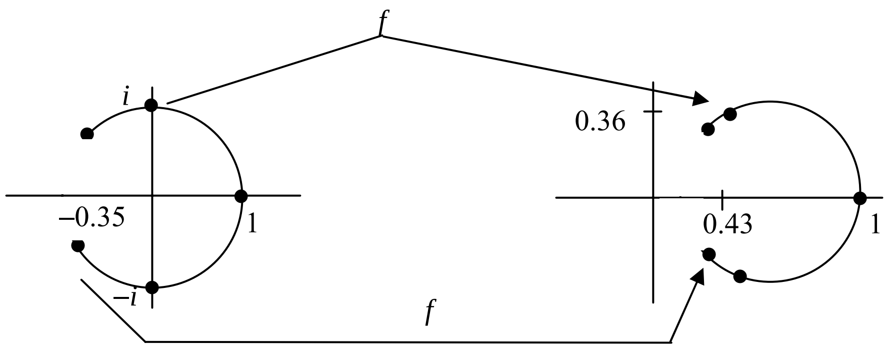

Figure 4

Unfortunately, the situation starts to get complicated now, for as *w* moves counterclockwise around the unit circle in the second quadrant with real part between approximately -0.38 and -0.45, the convergence of the sequence becomes exceedingly slow, and approaches the situation where *f*(*w*) may not even exist. Tens of thousands of terms are needed for the sequence to converge, with more terms needed when we move counterclockwise. Then when the real part of *w* is less than -0.45, divergence really sets in and the terms in the sequence oscillate wildly. For example, if *w* = -0.6 + 0.8*i* and the sequence $\{ z_n \}$ is defined by $z_1 = w,\ z_{n+1} = w^{z_n}$, then the first eight terms are as follows, and clearly no limit will exist.

n	z_n
1	-0.6 + 0.8i
2	0.04 - 0.165i
3	1.436 + 0.130i
4	-0.749 - 0.028i
5	-0.094 – 1.059i
6	10.218 – 2.155i
7	–95.063 – 69.951i
8	$-1.85\times10^{67} + 2.02\times10^{65}\, i$

One final comment concerns the graphical representation of the terms $w,\ w^w, \cdots$. If we plot these points in the plane, an interesting picture develops for the points get partitioned into three sets where each set forms a spiral that converges to the limit point $\alpha + i\beta = \gamma$ and the three spirals become more evenly spaced apart as $n \to \infty$. With the angle $\theta_n = \arg(z_{n+1} - \gamma) - \arg(z_n - \gamma)$, it follows that

$\theta_n \to \arg(\gamma) + \pi / 2$. The author in [1] has shown $\theta_n \to .719\pi$, or 129.4°, when $w = i$. In the case when $w = 0.6835 + 0.7299i$ the three spirals approach a limiting case of three rays (Figure 5) , each 120° apart from its neighbor. Points z_{10} through z_{15} are used to depict this situation.

n	z_n
1	w = (0.6835, 0.7299)
2	w^w
⋮	
10	(0.6405, 0.3731)
11	(0.6380, 0.3688)
12	(0.6410, 0.3688)
13	(0.6401, 0.3703)
14	(0.6396, 0.3694)
15	(0.6402, 0.3694)

The terms in this sequence converge to γ = 0.63999 + 0.36957i, and their graphic depiction is as follows.

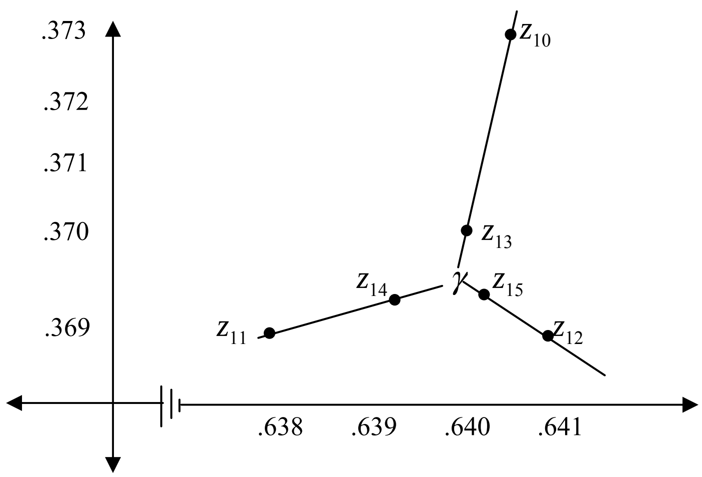

Figure 5

Students today have the opportunity to discover, and learn, some very good mathematics with not much more background material than elementary function theory and the technology available with the current graphing calculators or computer algebra systems.

References

Abbott, Steve and Greg Parker. Complex Power Iterations, *The Mathematical Gazette* Vol. 81, Nov. 1997.